A description of the quantum superalgebra $U_q[sl(n+1|m)]$ via creation and annihilation generators

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Abstract. A description of the quantum superalgebra $U_q[sl(n+1|m)]$ and in particular of the special linear superalgebra $sl(n+1|m)$ via creation and annihilation generators (CAGs) is given. It provides an alternative to the canonical description of $U_q[sl(n+1|m)]$ in terms of Chevalley generators. A conjecture that the Fock representations of the CAGs provide microscopic realizations of exclusion statistics is formulated.

1. Introduction

The description of the quantized simple (universal enveloping) Lie algebras [1, 2] and the basic Lie superalgebras [3-7] is usually carried out in terms of their Chevalley generators ($e_i, f_i, h_i,\ i = 1, \ldots, n$, for an algebra of rank n). Recently it has been pointed out that the quantum (super)algebras $U_q[osp(1|2n)]$ [8-10], $U_q[so(2n+1)]$ [11], more generally $U_q[osp(2r+1|2m)],\ r+m=n$ [12], and also $U_q[sl(n+1)]$ [13] can be defined via alternative sets of generators $a_i^{\pm},\ H_i,\ \ i = 1, \ldots, n$, referred to as (deformed) creation and annihilation generators (CAGs) or creation and annihilation operators.

The concept of creation and annihilation generators of a simple Lie (super)algebra was introduced in [14]. Let $\mathcal{A}$ be such an algebra with a supercommutator $[\![\ ,\ ]\!]$. The root vectors $a_1^{\xi}, \ldots, a_n^{\xi}$ of $\mathcal{A}$ are said to be creation ($\xi = +$) and annihilation ($\xi = -$) generators of $\mathcal{A}$, if

$$\mathcal{A} = lin.env.\{a_i^{\xi},\ [\![a_j^{\eta}, a_k^{\varepsilon}]\!] \mid i,j,k = 1, \ldots, n;\ \xi, \eta, \varepsilon = \pm\}, \tag{1}$$

so that $a_1^+, \ldots, a_n^+$ (resp. $a_1^-, \ldots, a_n^-$) are negative (resp. positive) root vectors of $\mathcal{A}$.

The justification for such terminology stems from the observation that the creation and the annihilation generators of the orthosymplectic Lie superalgebra (LS) $osp(2r+1|2m)$ have a direct physical significance: $a_1^{\pm}, \ldots, a_m^{\pm}$ (resp. $a_{m+1}^{\pm}, \ldots, a_n^{\pm}$) are para-Bose (resp. para-Fermi) operators [15], namely operators which generalize the statistics of the tensor (resp. spinor) fields in quantum field theory [16]. The LS $osp(2r+1|2m)$ is an algebra from the class B in the classification of Kac [17]. Therefore the paraquantizations (and hence the canonical Bose and Fermi quantization) could be called B-quantizations (or, more precisely, representations of a B-quantization).

A conjecture, stated in [18], assumes that to each class A, B, C and D of basic LSs [17] there corresponds a quantum statistics, so that its CAGs can be interpreted as creation and annihilation operators of real

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particles in the corresponding Fock space(s). This conjecture holds for the classes A, B, C and D of simple Lie algebras [19]. It was studied in more details for the Lie algebras $sl(n+1)$ ($A-$statistics) [20] and for the LSs $sl(1|m)$ ($A-$superstatistics) [14, 21]. As an illustration we mention that the Wigner quantum systems (WQSs), introduced in [22], are based on the $A-$superstatistics. These systems, which attracted some attention from different points of view [23-25], possess quite unconventional physical properties. For example, the $(n+1)-$particle WQS, based on the LS $sl(1|3n)$ [26], exhibits a quark like structure: the composite system occupies a small volume around the centre of mass and within it the geometry is noncommutative. The underlying statistics is a Haldane exclusion statistics [27], a subject of considerable interest in condensed matter physics.

We are not going to discuss further the properties of the superstatistics (for more details along this line see [28, 26] and the references therein). We mentioned this point here only in order to indicate that the alternative description of $sl(n+1|m)$ and $U_q[sl(n+1|m)]$ will be carried out in terms of (deformed) creation and annihilation generators, which, contrary to the Chevalley generators, could be of direct physical relevance too.

Throughout the paper we use the notation:

LS, LS's - Lie superalgebra, Lie superalgebras;

CAGs - creation and annihilation generators;

lin.env. - linear envelope;

$\mathbf{Z}$ - all integers;

$\mathbf{Z}_+$ - all non-negative integers;

$\mathbf{Z}_2=\{\bar{0},\bar{1}\}$ - the ring of all integers modulo 2;

$\mathbf{C}$ - all complex numbers;

$$[p;q]=\{p,p+1,p+2,\ldots,q-1,q\},\ \text{for } p\leq q\in\mathbf{Z}; \tag{2}$$

$$\theta_i=\begin{cases}\bar{0}, & \text{if } i=0,1,2,\ldots,n,\\ \bar{1}, & \text{if } i=n+1,n+2,\ldots,n+m,\end{cases};\quad \theta_{ij}=\theta_i+\theta_j; \tag{3}$$

$$[a,b]=ab-ba,\ \ \{a,b\}=ab+ba,\ \ [\![a,b]\!]=ab-(-1)^{deg(a)deg(b)}ba; \tag{4}$$

$$[a,b]_x=ab-xba,\ \ \{a,b\}_x=ab+xba,\ \ [\![a,b]\!]_x=ab-(-1)^{deg(a)deg(b)}xba. \tag{5}$$

2. The Lie superalgebra sl(n+1|m)

Here we give an alternative definition of the special linear Lie superalgebra $sl(n+1|m)$ in terms of creation and annihilation generators $a_1^\pm, a_2^\pm,\ldots,a_{n+m}^\pm$. We outline the relations between the CAGs and the Chevalley generators.

To begin with we recall that the universal enveloping algebra $U[gl(n+1|m)]$ of the general linear LS $gl(n+1|m)$ is a $\mathbf{Z}_2-$graded associative unital superalgebra generated by $(n+m+1)^2$ $\mathbf{Z}_2-$graded indeterminates $\{e_{ij}|i,j\in[0;n+m]\}$, $deg(e_{ij})=\theta_{ij}$, subject to the relations

$$[\![e_{ij},e_{kl}]\!]=\delta_{jk}e_{il}-(-1)^{\theta_{ij}\theta_{kl}}\delta_{il}e_{kj}\quad i,j,k,l\in[0;n+m]. \tag{6}$$

The LS $gl(n+1|m)$ is a subalgebra of $U[gl(n+1|m)]$, considered as a Lie superalgebra, with generators $\{e_{ij}|i,j\in[0;n+m]\}$ and supercommutation relations (6). The LS $sl(n+1|m)$ is a subalgebra of $gl(n+1|m)$:

$$sl(n+1|m)=lin.env.\{e_{ij},(-1)^{\theta_k}e_{kk}-(-1)^{\theta_l}e_{ll}|i\neq j;\ i,j,k,l\in[0;n+m]\}. \tag{7}$$

The generators $e_{00}, e_{11}, \ldots, e_{n+m,n+m}$ constitute a basis in the Cartan subalgebra of $gl(n+1|m)$. Denote by $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n+m}$ the dual basis, $\varepsilon_i(e_{jj}) = \delta_{ij}$. The root vectors of both $gl(n+1|m)$ and $sl(n+1|m)$ are $e_{ij},\ i \neq j,\ i,j \in [0; n+m]$. The root corresponding to e_{ij} is $\varepsilon_i - \varepsilon_j$. With respect to the natural order of the basis in the Cartan subalgebra e_{ij} is a positive (resp. a negative) root vector if $i < j$ (resp. $i > j$).

The above description of $sl(n+1|m)$ is simple, but it is not appropriate for quantum deformations. A more "economic" definition is given in terms of the Chevalley generators

$$\hat{h}_i = e_{i-1,i-1} - (-1)^{\theta_{i-1,i}} e_{ii}, \quad \hat{e}_i = e_{i-1,i}, \quad \hat{f}_i = e_{i,i-1}, \quad i \in [1; n+m] \tag{8}$$

and the $(n+m) \times (n+m)$ Cartan matrix $\{\alpha_{ij}\}$ with entries

$$\alpha_{ij} = (1 + (-1)^{\theta_{i-1,i}})\delta_{ij} - (-1)^{\theta_{i-1,i}}\delta_{i,j-1} - \delta_{i-1,j}, \quad i,j \in [1; n+m]. \tag{9}$$

We are working with a nonsymmetric Cartan matrix [17]. For instance the Cartan matrix (9), corresponding to $n+1 = 3,\ \ m = 5$ is 7×7 dimensional matrix:

$$(\alpha_{ij}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{10}$$

$U[sl(n{+}1|m)]$ is an associative unital algebra of the Chevalley generators, subject to the Cartan-Kac relations

$$[\hat{h}_i, \hat{h}_j] = 0, \quad [\hat{h}_i, \hat{e}_j] = \alpha_{ij}\hat{e}_j, \quad [\hat{h}_i, \hat{f}_j] = -\alpha_{ij}\hat{f}_j, \quad [\![\hat{e}_i, \hat{f}_j]\!] = \delta_{ij}\hat{h}_i, \tag{11}$$

and the Serre relations

$$[\hat{e}_i, \hat{e}_j] = 0, \quad [\hat{f}_i, \hat{f}_j] = 0, \quad if\ |i-j| \neq 1; \tag{12a}$$

$$\hat{e}_{n+1}^2 = 0, \quad \hat{f}_{n+1}^2 = 0; \tag{12b}$$

$$[\hat{e}_i, [\hat{e}_i, \hat{e}_{i+1}]] = 0, \quad [\hat{f}_i, [\hat{f}_i, \hat{f}_{i+1}]] = 0, \quad i \neq n+m; \tag{12c}$$

$$[\hat{e}_{i+1}, [\hat{e}_{i+1}, \hat{e}_i]] = 0, \quad [\hat{f}_{i+1}, [\hat{f}_{i+1}, \hat{f}_i]] = 0, \quad i \neq n+m; \tag{12d}$$

$$\{[\hat{e}_{n+1}, \hat{e}_n], [\hat{e}_{n+1}, \hat{e}_{n+2}]\} = 0, \quad \{[\hat{f}_{n+1}, \hat{f}_n], [\hat{f}_{n+1}, \hat{f}_{n+2}]\} = 0. \tag{12e}$$

The so-called additional Serre relations ($12e$) [29, 30, 31] can be written also in the form

$$\{\hat{e}_{n+1}, [[\hat{e}_n, \hat{e}_{n+1}], \hat{e}_{n+2}]\} = 0, \quad \{\hat{f}_{n+1}, [[\hat{f}_n, \hat{f}_{n+1}], \hat{f}_{n+2}]\} = 0. \tag{12f}$$

The grading on $U[sl(n+1|m)]$ is induced from the requirement that the only odd generators are $\hat{e}_{n+1}$ and $\hat{f}_{n+1}$, namely

$$deg(\hat{h}_i) = \hat{0}, \quad deg(\hat{e}_i) = deg(\hat{f}_i) = \theta_{i-1,i}. \tag{13}$$

The LS $sl(n+1|m)$ is a subalgebra of $U[sl(n+1|m)]$, generated by the Chevalley generators in a sense of a Lie superalgebra. It is a linear span of the Chevalley generators (8) and all root vectors

$$\begin{aligned} e_{ij} &= [[[\ldots[[\hat{e}_{i+1},\hat{e}_{i+2}],\hat{e}_{i+3}],\ldots],\hat{e}_{j-1}],\hat{e}_j], \\ e_{ji} &= [\hat{f}_j,[\hat{f}_{j-1},[\ldots,[\hat{f}_{i+2},\hat{f}_{i+1}]\ldots,]]], \quad i+1<j;\ i,j\in[0;n+m]. \end{aligned} \tag{14}$$

Consider the following root vectors from $sl(n+1|m)$:

$$\hat{a}_i^+ = e_{i0}, \quad \hat{a}_i^- = e_{0i}, \ \ i\in[1;n+m], \tag{15}$$

or, equivalently from (14)

$$\hat{a}_1^- = \hat{e}_1, \quad \hat{a}_i^- = [[[\ldots[[\hat{e}_1,\hat{e}_2],\hat{e}_3],\ldots],\hat{e}_{i-1}],\hat{e}_i] = [\hat{a}_{i-1}^-,e_i], \quad i\in[2;n+m], \tag{16a}$$

$$\hat{a}_1^+ = \hat{f}_1, \quad \hat{a}_i^+ = [\hat{f}_i,[\hat{f}_{i-1},[\ldots,[\hat{f}_3,[\hat{f}_2,\hat{f}_1]]\ldots]]] = [f_i,\hat{a}_{i-1}^+]. \quad i\in[2;n+m]. \tag{16b}$$

The root of a_i^- (resp. of a_i^+) is $\varepsilon_0-\varepsilon_i$ (resp. $\varepsilon_i-\varepsilon_0$). Therefore (with respect to the natural order of the basis $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{n+m}$) $a_1^-,\ldots,a_{n+m}^-$ are positive root vectors, whereas $a_1^+,\ldots,a_{n+m}^+$ are negative root vectors. Moreover, Eq. (1) with $\mathcal{A}=sl(n+1|m)$ holds. Hence, the generators (15) are creation and annihilation generators of $sl(n+1|m)$. These generators satisfy the following triple relations:

$$[\![\hat{a}_i^\xi,\hat{a}_j^\xi]\!]=0, \quad \xi=\pm, \quad i,j=1,2,\ldots,n+m, \tag{17a}$$

$$[\![[\![\hat{a}_i^+,\hat{a}_j^-]\!],\hat{a}_k^+]\!]=\delta_{jk}\hat{a}_i^+ + (-1)^{\theta_i}\delta_{ij}\hat{a}_k^+, \quad i,j,k=1,2,\ldots,n+m, \tag{17b}$$

$$[\![[\![\hat{a}_i^+,\hat{a}_j^-]\!],\hat{a}_k^-]\!]=-(-1)^{\theta_{ij}\theta_k}\delta_{ik}\hat{a}_j^- - (-1)^{\theta_i}\delta_{ij}\hat{a}_k^-, \quad i,j,k\in[1;n+m]. \tag{17c}$$

The CAGs (15) together with (17) define completely $sl(n+1|m)$. The relations (17) are however (similar as Eqs. (6)) not convenient for quantization. It turns out, and this is a new result, that one can take only a part of the relations (17), so that they still define completely $sl(n+1|m)$ and, as we shall see, are appropriate for Hopf algebra deformations.

Proposition 1. $U[sl(n+1|m)]$ is an associative unital superalgebra with generators $\hat{a}_i^\pm$, $i\in[1;n+m]$ and relations:

$$[\![\hat{a}_1^\xi,\hat{a}_2^\xi]\!]=0, \quad [\![a_1^\xi,a_1^\xi]\!]=0, \quad \xi=\pm, \tag{18a}$$

$$[\![[\![\hat{a}_i^+,\hat{a}_j^-]\!],\hat{a}_k^+]\!]=\delta_{jk}\hat{a}_i^+ + (-1)^{\theta_i}\delta_{ij}\hat{a}_k^+, \quad |i-j|\le 1, \ \ i,j,k\in[1;n+m], \tag{18b}$$

$$[\![[\![\hat{a}_i^+,\hat{a}_j^-]\!],\hat{a}_k^-]\!]=-(-1)^{\theta_{ij}\theta_k}\delta_{ik}\hat{a}_j^- - (-1)^{\theta_i}\delta_{ij}\hat{a}_k^-, \ ;|i-j|\le 1, \ i,j,k\in[1;n+m] \tag{18c}$$

The $\mathbf{Z}_2$−grading in $U[sl(n+1|m)]$ is induced from

$$deg(\hat{a}_i^\pm)=\theta_i. \tag{19}$$

The proof follows from the expressions of the Chevalley generators (8) via the CAGs:

$$\hat{h}_1=[\![\hat{a}_1^-,\hat{a}_1^+]\!], \quad \hat{h}_i=(-1)^{\theta_{i-1}}([\![\hat{a}_i^-,\hat{a}_i^+]\!]-[\![\hat{a}_{i-1}^-,\hat{a}_{i-1}^+]\!]), \quad i\in[2;n+m], \tag{20a}$$

$$\hat{e}_1=\hat{a}_1^-, \quad \hat{f}_1=\hat{a}_1^+, \quad \hat{e}_i=[\![\hat{a}_{i-1}^+,\hat{a}_i^-]\!], \quad \hat{f}_i=[\![\hat{a}_i^+,\hat{a}_{i-1}^-]\!]. \quad i\in[2;n+m]. \tag{20b}$$

We skip the proof of Eqs. (20), since we will give a detailed proof in the quantum case (see the *Theorem*). Only from (18) one derives also the larger set of relation (17).

3. Description of $\mathbf{U_q[sl(n+1|m)]}$ via deformed CAGs

In this section we define the quantum superalgebra $U_q[sl(n+1|m)]$ in terms of deformed creation and annihilation generators $a_i^\pm, H_i, \;\; i=1,2,\ldots,n+m$. The CAGs are elements from the so-called Cartan-Weyl basis of $U_q[sl(n+1|m)]$. A general procedure to construct such a basis was given in [7] (see also [29]). We follow this procedure and identify the deformed $a_1^\pm,\ldots,a_{n+m}^\pm$ generators with those elements of the Cartan-Weyl basis, which reduce to the nondeformed CAGs (16) in the limit $q\to 1$.

First we introduce $U_q[sl(n+1|m)]$ by means of its classical definition in terms of the Cartan matrix (9) and the Chevalley generators. Let $\mathbf{C}[[h]]$ be the complex algebra of formal power series in the indeterminate h, $q=e^h\in \mathbf{C}[[h]]$. $U_q[sl(n+1|m)]$ is a Hopf algebra, which is a topologically free $\mathbf{C}[[h]]$ module (complete in the $h-$adic topology), with (Chevalley) generators $\{h_i,e_i,f_i\}_{i\in[1;n+m]}$ subject to the Cartan-Kac relations $(\bar q = q^{-1})$

$$[h_i,h_j]=0, \tag{21a}$$

$$[h_i,e_j]=\alpha_{ij}e_j, \quad [h_i,f_j]=-\alpha_{ij}f_j, \tag{21b}$$

$$[\![e_i,f_j]\!]=\delta_{ij}\frac{k_i-\bar k_i}{q-\bar q}, \quad k_i=q^{h_i},\; k_i^{-1}\equiv \bar k_i = q^{-h_i}, \tag{21c}$$

the e-Serre relations (see (5))

$$[e_i,e_j]=0,\; if\; |i-j|\neq 1; \quad e_{n+1}^2=0; \tag{22a}$$

$$[e_i,[e_i,e_{i\pm1}]_{\bar q}]_q=[e_i,[e_i,e_{i\pm1}]_q]_{\bar q}=0, \quad i\neq n+1, \tag{22b}$$

$$\{e_{n+1},[[e_n,e_{n+1}]_q,e_{n+2}]_{\bar q}\}=\{e_{n+1},[[e_n,e_{n+1}]_{\bar q},e_{n+2}]_q\}=0, \tag{22c}$$

and the $f-$Serre relations, obtained from the e-Serre relations by replacing everywhere e_i with f_i:

$$[f_i,f_j]=0,\; if\; |i-j|\neq 1; \quad f_{n+1}^2=0; \tag{22d}$$

$$[f_i,[f_i,f_{i\pm1}]_{\bar q}]_q=[f_i,[f_i,f_{i\pm1}]_q]_{\bar q}=0, \quad i\neq n+1, \tag{22e}$$

$$\{f_{n+1},[[f_n,f_{n+1}]_q,f_{n+2}]_{\bar q}\}=\{f_{n+1},[[f_n,f_{n+1}]_{\bar q},f_{n+2}]_q\}=0. \tag{22f}$$

From (21b) one derives the following useful relations:

$$k_ie_j=q^{\alpha_{ij}}e_jk_i, \quad k_if_j=q^{-\alpha_{ij}}f_jk_i, \quad \bar k_ie_j=q^{-\alpha_{ij}}e_j\bar k_i, \quad \bar k_if_j=q^{\alpha_{ij}}f_j\bar k_i. \tag{23}$$

We do not write the other Hopf algebra maps $(\Delta,\;\varepsilon,S)$ (see [7, 29]), since we will not use them. They are certainly also a part of the definition.

Remark. We consider h as an indeterminate. All relations remain also true, if one replaces h with a number, so that q is not a root of 1. The latter corresponds to a transition from $U_q[sl(n+1|m)$ to the factor algebra $U_q[sl(n+1|m)]/h=number$.

Following [7, 29], introduce a normal order in the system of the positive roots $\Delta_+ = \{\varepsilon_i - \varepsilon_j | i < j \in [0; n+m]\}$ as follows:

$$\varepsilon_i - \varepsilon_j < \varepsilon_k - \varepsilon_l \ \text{ if } \ j < l \ \text{ or if } \ j = l \ \text{ and } \ i < k.$$

Taking into account Eqs. (16), we define the deformed CAGs to be Cartan-Weyl basis vectors, which are in agreement with the above normal order:

$$a_1^- = e_1, \quad a_i^- = [[[\ldots[[e_1, e_2]_{\bar{q}_1}, e_3]_{\bar{q}_2}, \ldots]_{\bar{q}_{i-3}}, e_{i-1}]_{\bar{q}_{i-2}}, e_i]_{\bar{q}_{i-1}} = [a_{i-1}^-, e_i]_{\bar{q}_{i-1}}, \tag{24a}$$

$$a_1^+ = f_1, \quad a_i^+ = [f_i, [f_{i-1}, [\ldots, [f_3, [f_2, f_1]_{q_1}]_{q_2} \ldots]_{q_{i-3}}]_{q_{i-2}}]_{q_{i-1}} = [f_i, a_{i-1}^+]_{q_{i-1}}, \tag{24b}$$

$$H_1 = h_1, \quad H_i = h_1 + (-1)^{\theta_1} h_2 + (-1)^{\theta_2} h_3 + \ldots + (-1)^{\theta_{i-1}} h_i, \tag{24c}$$

where

$$q_i = q^{1-2\theta_i} = \begin{cases} q, & \text{if } i \leq n, \\ \bar{q}, & \text{if } i > n. \end{cases} \tag{25}$$

Note that Eqs. (21)-(23) are invariant with respect to the antilinear antiinvolution $(\ )^*$, defined as

$$(h)^* = -h, \quad (h_i)^* = h_i, \quad (e_i)^* = f_i, \quad (f_i)^* = e_i, \quad (ab)^* = (b)^*(a)^*. \tag{26}$$

Therefore

$$(q)^* = \bar{q}, \quad (k_i)^* = \bar{k_i}, \quad (a_i^{\pm})^* = a_i^{\mp}, \quad (H_i)^* = H_i. \tag{27}$$

The next proposition will be used in several intermediate computations.

Proposition 2. The relations $(i \neq 1)$

$$[\![e_i, a_j^-]\!]_{q_j^{\delta_{i-1,j}-\delta_{ij}}} = -q_{i-1}\delta_{i-1,j} a_i^-, \tag{28a}$$

$$[\![f_i, a_j^+]\!]_{q_j^{\delta_{i-1,j}-\delta_{ij}}} = \delta_{i-1,j} a_i^+, \tag{28b}$$

$$[\![e_i, a_j^+]\!] = \delta_{ij} a_{i-1}^+ k_i^{-(-1)^{\theta_{i-1}}}, \tag{28c}$$

$$[\![f_i, a_j^-]\!] = -(-1)^{\theta_{i-1,i}} \delta_{ij} k_i^{(-1)^{\theta_{i-1}}} a_{i-1}^-. \tag{28d}$$

follow from (21)-(23) and the definition of the CAGs (24).

Proof.

A) Consider first (28a).

(i) The case $j < i-1$. Eq. (28a) is an immediate consequence of (22a).

(ii) The case $j = i-1$ reduces to the definition (24a).

(iii) The case $j = i$.

(iii.1) $i = 2$.

(iii.1a) If $n = 0$, $[\![e_2, a_2^-]\!]_{\bar{q}_2} = [e_2, [e_1, e_2]_q]_q = -q[e_2, [e_2, e_1]_{\bar{q}}]_q = 0$, according to (22$b$).

(iii.1b) If $n = 1$, $[\![e_2, a_2^-]\!]_{\bar{q}_2} = \{e_2, a_2^-\}_{\bar{q}_2} = \{e_2, [e_1, e_2]_{\bar{q}}\}_q$
$= e_2 e_1 e_2 + q e_1 e_2^2 - \bar{q} e_2^2 e_1 - \bar{q} q e_2 e_1 e_2 = 0$ since, see (22a), $e_2^2 = 0$.

(iii.1c) If $n > 1$, $[\![e_2, a_2^-]\!]_{\bar{q}_2} = [e_2, [e_1, e_2]_{\bar{q}_1}]_{\bar{q}_2} = -\bar{q}[e_2, [e_2, e_1]_q]_{\bar{q}} = 0$ (see (22b).

(iii.2) $i > 2$. Using the identity

$$\text{If} \quad [\![a,b]\!]=0, \ \ \text{then} \ \ [\![[\![a,c]\!]_q,b]\!]_p=[\![a,[\![c,b]\!]_p]\!]_q, \quad p,q\in \mathbf{C}[[h]], \tag{29}$$

and the circumstance that $[e_i,a^-_{i-2}]=0$, one obtains from (24a)
$a^-_i=[[a^-_{i-2},e_{i-1}]_{\bar{q}_{i-2}},e_i]_{\bar{q}_{i-1}}=[a^-_{i-2},[e_{i-1},e_i]_{\bar{q}_{i-1}}]_{\bar{q}_{i-2}}$.

(iii.2a) If $i=n+1$,
$[\![e_{n+1},a^-_{n+1}]\!]_{\bar{q}_{n+1}}=\{e_{n+1},[a^-_{n-1},[e_n,e_{n+1}]_{\bar{q}_n}]_{\bar{q}_{n-1}}\}_{\bar{q}_{n+1}}=\{e_{n+1},[a^-_{n-1},[e_n,e_{n+1}]_{\bar{q}}]_{\bar{q}}\}_q$.
Set $a=e_{n+1}$, $b=a^-_{n-1}$, $c=[e_n,e_{n+1}]_{\bar{q}}$; take into account that $[\![a,b]\!]=0$ and apply the identity

$$\text{If} \ \ [\![a,b]\!]=0, \quad [\![a,[\![b,c]\!]_q]\!]_p=(-1)^{\alpha\beta}[\![b,[\![a,c]\!]_p]\!]_q, \ \ \alpha=deg(a), \ \ \beta=deg(b). \tag{30}$$

Then $[\![e_{n+1},a^-_{n+1}]\!]_{\bar{q}_{n+1}}=[a^-_{n-1},z]_{\bar{q}}$=0, since $z=\{e_{n+1},[e_n,e_{n+1}]_{\bar{q}}\}_q=0$ (follows from $e^2_{n+1}=0$).

(iii.2b) If $i\neq n+1$, then $y=[e_i,[e_i,e_{i-1}]_{q_{i-1}}]_{\bar{q}_i}$=0, since in both cases $i\leq n$ or $i>n+1$ it reduces to (22b).
Therefore, $[\![e_i,a^-_i]\!]_{\bar{q}_i}=[e_i,a^-_i]_{\bar{q}_i}=[e_i,[a^-_{i-2},[e_{i-1},e_i]_{\bar{q}_{i-1}}]_{\bar{q}_{i-2}}]_{\bar{q}_i}$
(if $a=e_i$, $\ b=a^-_{i-2}$, $\ c=[e_{i-1},e_i]_{\bar{q}_{i-1}}$ then $[a,b]=0$ and from (30))
$=[a^-_{i-2},[e_i,[e_{i-1},e_i]_{\bar{q}_{i-1}}]_{\bar{q}_i}]_{\bar{q}_{i-2}}=-\bar{q}_{i-1}[a^-_{i-2},y]_{\bar{q}_{i-2}}=0$. Hence (28a) holds for any $i=j>1$.

(iv) The case $j=i+1$.

(iv.1) If $i=2$, $\ n+1\neq 2$, $[\![e_2,a^-_3]\!]=[e_2,[\ [e_1,e_2]_{\bar{q}_1},e_3]_{\bar{q}_2}]=[e_2,[\ [e_1,e_2]_{\bar{q}_1},e_3]_{\bar{q}_1}]$. For $b=e_2$, $\ a=e_1$, $\ c=e_3$ use the identity:

If b is even and $[\![a,c]\!]=0$, then

$$(x+\bar{x})[b,[\![a,[b,c]_x]\!]_x]=[\![a,[b,[b,c]_x]_{\bar{x}}]\!]_{x^2}-[\![[b,[b,a]_x]_{\bar{x}},c]\!]_{x^2}, \quad \bar{x}=x^{-1}. \tag{31}$$

Then $[\![e_2,a^-_3]\!]=(q_1+\bar{q}_1)^{-1}\left([\![e_1,[e_2,[e_2,e_3]_{\bar{q}_1}]_{q_1}]\!]_{q_1^{-2}}-[\![[e_2,[e_2,e_1]_{\bar{q}_1}]_{q_1},e_3]\!]_{q_1^{-2}}\right)=0$ according to (22b).

(iv.2) If $i=2$, $\ n+1=2$, $\ [\![e_2,a^-_3]\!]=\{e_2,[[e_1,e_2]_{\bar{q}},e_3]_q\}=0$ according to (22c).

(iv.3) For $i>2$, set (see (24a)) $a^-_{i+1}=[[[a^-_{i-2},e_{i-1}]_{\bar{q}_{i-2}},e_i]_{\bar{q}_{i-1}},e_{i+1}]_{\bar{q}_i}$. Use that $[a^-_{i-2},e_i]=[a^-_{i-2},e_{i+1}]=0$ and apply twice (29): $a^-_{i+1}=[a^-_{i-2},[[e_{i-1},e_i]_{\bar{q}_{i-1}},e_{i+1}]_{\bar{q}_i}]_{\bar{q}_{i-2}}$.

(iv.3a) If $i=n+1$, $[\![e_i,a^-_{i+1}]\!]=\{e_{n+1},a^-_{n+2}\}=\{e_{n+1},[a^-_{n-1},[[e_n,e_{n+1}]_{\bar{q}_n},e_{n+2}]_{\bar{q}_{n+1}}]_{\bar{q}_{n-1}}\}$
(use that $[e_{n+1},a^-_{n-1}]=0$ and (30))
$=[a^-_{n-1},\{e_{n+1},[[e_n,e_{n+1}]_{\bar{q}_n},e_{n+2}]_{\bar{q}_{n+1}}\}]_{\bar{q}_{n-1}}=0$ according to (22c) and (25).

(iv.3b) If $i\neq n+1$ $\ \ [\![e_i,a^-_{i+1}]\!]=[e_i,a^-_{i+1}]=[e_i,[a^-_{i-2},[[e_{i-1},e_i]_{\bar{q}_{i-1}},e_{i+1}]_{\bar{q}_i}]_{\bar{q}_{i-2}}]$
($[e_i,a^-_{i-2}]=0$, use (30))
$=[a^-_{i-2},[e_i,[[e_{i-1},e_i]_{\bar{q}_{i-1}},e_{i+1}]_{\bar{q}_i}]]_{\bar{q}_{i-2}}=[a^-_{i-2},[e_i,[[e_{i-1},e_i]_{\bar{q}_i},e_{i+1}]_{\bar{q}_i}]]_{\bar{q}_{i-2}}$.
If $a=e_i$, $b=e_{i-1}$, $c=e_{i+1}$, then $[b,c]=0$; apply a similar to (31) identity:

If a is even and $[\![b,c]\!]=0$, then

$$(x+\bar{x})[a,[\![[b,a]_x,c]\!]_x]=[\![b,[a,[a,c]_x]_{\bar{x}}]\!]_{x^2}-[\![[a,[a,b]_x]_{\bar{x}},c]\!]_{x^2}. \tag{32}$$

The latter yields $[\![e_i,a^-_{i+1}]\!]$
$=(\bar{q}_i+q_i)^{-1}[a^-_{i-2},\left([e_{i-1},[e_i,[e_i,e_{i+1}]_{\bar{q}_i}]_{q_i}]_{q_i^{-2}}-[[e_i,[e_i,e_{i-1}]_{\bar{q}_i}]_{q_i},e_{i+1}]_{q_i^{-2}}\right)]_{\bar{q}_{i-2}}=0$ according to (22b).

(v) The case $j > i+1$. Then $a_j^- = [[[\ldots[[a_{i+1}^-, e_{i+2}]_{\bar{q}_{i+1}}, e_{i+3}]_{\bar{q}_{i+2}}, \ldots]_{\bar{q}_{j-3}}, e_{j-1}]_{\bar{q}_{j-2}}, e_j]_{\bar{q}_{j-1}}$ and since e_i commutes with $e_{i+2},\ \ e_{i+3},\ \ \ldots, e_j$, see (22a), and e_i supercommutes with a_{i+1}^-, see (iv), one concludes that $[\![e_i, a_j^-]\!] = 0$. The unification of (i)-(v) yields (28a).

B) Applying the antiinvolution (26) on both sides of (28a) one obtains (28b).

C) We pass to prove (28c).

(i) For $i > j$, (28c) is an immediate consequence of (24b) and (21c).

(ii) Let $i = j$. $[\![e_i, a_i^+]\!] = [\![e_i, [f_i, a_{i-1}^+]_{q_{i-1}}]\!]$
(from (i) $[\![e_i, a_{i-1}^+]\!] = 0$, apply (29))
$= [\![[\![e_i, f_i]\!], a_{i-1}^+]\!]_{q_{i-1}} = [\frac{k_i - \bar{k}_i}{q-\bar{q}}, a_{i-1}^+]_{q_{i-1}} = a_{i-1}^+ k_i^{-(-1)^{\theta_{i-1}}}$. In the last step we used the relations $k_i a_{i-1}^+ = q a_{i-1}^+ k_i$ and $\bar{k}_i a_{i-1}^+ = \bar{q} a_{i-1}^+ \bar{k}_i$, which follow from (24$b$) and (23).

(iii) Let $j = i+1$. $[\![e_i, a_{i+1}^+]\!] = [\![e_i, [f_{i+1}, [f_i, a_{i-1}^+]_{q_{i-1}}]_{q_i}]\!]$
(take into account that $[e_i, f_{i+1}] = 0$ and apply (30))
$= [\![f_{i+1}, [\![e_i, [f_i, a_{i-1}^+]_{q_{i-1}}]\!]]\!]_{q_i}$ (now $[e_i, a_{i-1}^+] = 0$, use (29))
$= [\![f_{i+1}, [\![[\![e_i, f_i]\!], a_{i-1}^+]\!]_{q_{i-1}}]\!]_{q_i} = [\![f_{i+1}, [\![\frac{k_i - \bar{k}_i}{q-\bar{q}}, a_{i-1}^+]\!]_{q_{i-1}}]\!]_{q_i} = [f_{i+1}, a_{i-1}^+ k^{-(-1)^{\theta_{i-1}}}]_{q_i}$
Using the identity

$$[a, bc]_x = [a, b]c + b[a, c]_x \tag{33}$$

one has $[\![e_i, a_{i+1}^+]\!] = [f_{i+1}, a_{i-1}^+] k_i^{-(-1)^{\theta_{i-1}}} + a_{i-1}^+ [f_{i+1}, k_i^{-(-1)^{\theta_{i-1}}}]_{q_i} = 0$, according to (28$b$), (23) and (25).

(iv) For $j > i+1$ $\ a_j^+ = [f_j, [f_{j-1}, [\ldots, [f_{i+3}, [f_{i+2}, a_{i+1}^+]_{q_{i+1}}]_{q_{i+2}} \ldots]_{q_{j-3}}]_{q_{j-2}}]_{q_{j-1}}$
and since e_i supercommutes with a_{i+1}^+, see (iii), and commutes with $f_j,\ \ f_{j-1},\ \ \ldots, f_{i+2}$, see (21$c$), one concludes that $[\![e_i, a_j^+]\!] = 0$. The unification of (i)-(iv) yields (28c).

D) Applying the antiinvolution (26) on both sides of (28c) one obtains (28d).

This completes the proof.

Proposition 3. The deformed CAG's (23) generate $U_q[sl(n+1|m)]$.

Proof. Let

$$L_i = q^{H_i}, \quad \bar{L}_i \equiv L_i^{-1} = q^{-H_i}. \tag{34}$$

The proof is a consequence of the relations

$$[\![a_i^-, a_i^+]\!] = \frac{L_i - \bar{L}_i}{q - \bar{q}} \tag{35a}$$

$$[\![a_i^-, a_{i+1}^+]\!] = -(-1)^{\theta_i} L_i f_{i+1} \tag{35b}$$

$$[\![a_{i+1}^-, a_i^+]\!] = -(-1)^{\theta_i} e_{i+1} \bar{L}_i \tag{35c}$$

We prove these equations by induction on i. For $i = 1$, (35a) holds. Let (35a) be true. Then from (28d), (30) and (35a) one has

$[\![a_i^-, a_{i+1}^+]\!] = [\![a_i^-, [f_{i+1}, a_i^+]_{q_i}]\!] = [\![f_{i+1}, [\![a_i^-, a_i^+]\!]]_{q_i} = \frac{1}{q-\bar{q}}[f_{i+1}, L_i - \bar{L}_i]_{q_i}$

$= \frac{1}{q-\bar{q}}[f_{i+1}, k_1 k_2^{(-1)^{\theta_1}} k_3^{(-1)^{\theta_2}} \dots k_i^{(-1)^{\theta_{i-1}}} - \bar{k}_1 k_2^{-(-1)^{\theta_1}} k_3^{-(-1)^{\theta_2}} \dots k_i^{-(-1)^{\theta_{i-1}}}]_{q_i}.$

Using (25) and repeatedly (23), one end with

$[\![a_i^-, a_{i+1}^+]\!] = -(-1)^{\theta_i} k_1 k_2^{(-1)^{\theta_1}} k_3^{(-1)^{\theta_2}} \dots k_i^{(-1)^{\theta_{i-1}}} f_{i+1}$, namely with $(35b)$. Similarly, one proves $(35c)$. Therefore, if $(35a)$ holds, then also equations $(35b)$ and $(35c)$ are fulfilled. Assuming this, consider $[\![a_{i+1}^-, a_{i+1}^+]\!] = [\![[a_i^-, e_{i+1}]_{\bar{q}_i}, a_{i+1}^+]\!]$. Then the identity

$$[\![[\![a, b]\!]_x, c]\!] = (-1)^{\beta\gamma}[\![[\![a, c]\!], b]\!]_x + [\![a, [\![b, c]\!]]\!]_x, \quad \beta = deg(b),\ \gamma = deg(c) \tag{36}$$

yields

$$[\![a_{i+1}^-, a_{i+1}^+]\!] = (-1)^{\theta_{i,i+1}}[\![[\![a_i^-, a_{i+1}^+]\!], e_{i+1}]\!]_{\bar{q}_i} + [\![a_i^-, [\![e_{i+1}, a_{i+1}^+]\!]]\!]_{\bar{q}_i}$$

$$= -(-1)^{\theta_{i+1}}[k_1 k_2^{(-1)^{\theta_1}} k_3^{(-1)^{\theta_2}} \dots k_i^{(-1)^{\theta_{i-1}}} f_{i+1}, e_{i+1}]_{\bar{q}_i} + [\![a_i^-, a_i^+ k_{i+1}^{-(-1)^{\theta_i}}]\!]_{\bar{q}_i}$$

$$= -(-1)^{\theta_{i+1}}[\![f_{i+1}, e_{i+1}]\!] k_1 k_2^{(-1)^{\theta_1}} k_3^{(-1)^{\theta_2}} \dots k_i^{(-1)^{\theta_{i-1}}} + [\![a_i^-, a_i^+]\!] k_{i+1}^{-(-1)^{\theta_i}}$$

$$= \frac{k_1 k_2^{(-1)^{\theta_1}} k_3^{(-1)^{\theta_2}} \dots k_i^{(-1)^{\theta_{i-1}}} k_{i+1}^{(-1)^{\theta_i}} - \bar{k}_1 k_2^{-(-1)^{\theta_1}} k_3^{-(-1)^{\theta_2}} \dots k_i^{-(-1)^{\theta_{i-1}}} k_{i+1}^{-(-1)^{\theta_i}}}{q - \bar{q}}$$

$$= \frac{L_{i+1} - \bar{L}_{i+1}}{q - \bar{q}}.$$

Thus, Eqs (35) hold for any i. From (24c) and (35) we have

$$e_1 = a_1^-, \quad e_{i+1} = -(-1)^{\theta_i}[\![a_{i+1}^-, a_i^+]\!] q^{H_i}, \quad i \in [1; n+m-1] \tag{37a}$$

$$f_1 = a_1^+, \quad f_{i+1} = -(-1)^{\theta_i}\bar{q}^{H_i}[\![a_i^-, a_{i+1}^+]\!], \quad i \in [1; n+m-1] \tag{37b}$$

$$h_1 = H_1, \quad h_i = (-1)^{\theta_{i-1}}(H_i - H_{i-1}) \quad i \in [2; n+m], \tag{37c}$$

which completes the proof.

We proceed to state our main result.

Theorem. $U_q[sl(n+1|m)]$ is an unital associative algebra, which is topologically free $\mathbf{C}[[h]]$ module, with generators $\{H_i,\ a_i^{\pm}\}_{i\in[1;n+m]}$ and relations

$$[H_i, H_j] = 0, \tag{38a}$$

$$[H_i, a_j^{\pm}] = \mp(1 + (-1)^{\theta_i}\delta_{ij}) a_j^{\pm}, \tag{38b}$$

$$[\![a_i^-, a_i^+]\!] = \frac{L_i - \bar{L}_i}{q - \bar{q}}, \tag{38c}$$

$$[\![[\![a_i^{\eta}, a_{i+\xi}^{-\eta}]\!], a_k^{\eta}]\!]_{q^{\xi(1+(-1)^{\theta_i}\delta_{ik})}} = \eta^{\theta_k}\delta_{k,i+\xi} L_k^{-\xi\eta} a_i^{\eta}, \quad \xi, \eta = \pm \ \ or \ \ \pm 1, \tag{38d}$$

$$[\![a_1^{\xi}, a_2^{\xi}]\!]_q = 0, \quad [\![a_1^{\xi}, a_1^{\xi}]\!] = 0, \quad \xi = \pm. \tag{38e}$$

Proof. As a first step one has to show that Eqs. (38) hold. Most of the results for this part of the proof are already obtained. Eq. (38a) is evident. Eq. (38b) follows from the relation $\sum_{p=1}^{i}\sum_{q=1}^{j}(-1)^{\theta_{p-1}}\alpha_{pq} = 1+(-1)^{\theta_i}\delta_{ij}$, the definitions of $a_i^{\pm}$ and H_i (see (24)) and the relations (21b). From (38b) one also derives

$$L_i a_j^{\pm} = q^{\mp(1+(-1)^{\theta_i}\delta_{ij})} a_j^{\pm} L_i. \qquad (39)$$

Eq. (38c) is the same as (35a). The derivation of all triple relations (38d) is relatively long, but simple. It is based on a case by case considerations. To this end one replaces e_i and f_i in (28) with the right hand sides of (37a, b). The nontrivial part is to put all cases in the compact form (38d). If $n \neq 0$, $[\![a_1^{\xi}, a_1^{\xi}]\!] = [a_1^{\xi}, a_1^{\xi}] = 0$. The first relations in (38e) reduce to the triple Serre relations (22b, e). If $n = 0$, Eqs. (38e) hold because $e_1^2 = 0$ and $f_1^2 = 0$.

It remains to prove as a second step that any other relation in $U_q[sl(n+1|m)]$ follows from Eqs. (38). To this end it suffices to show that all Cartan-Kac relations (21) and the Serre relations (22) follow from (38).

A) The Cartan-Kac relations (21a) follow in an evident way from (37c) and (38a).

B) Eqs. (21b) are easily derived from (37) and (38b).

C) The proof of (21c) is not trivial.

(i) The case $i = j = 1$ is evident.

(ii) The case $i = 1,\ j > 1$: $[\![f_j, e_1]\!] = [\![-(-1)^{\theta_{j-1}}\bar{L}_{j-1}[\![a_{j-1}^-, a_j^+]\!], a_1^-]\!]$

(using (39))

$= -(-1)^{\theta_{j-1}}\bar{L}_{j-1}[\![[\![a_{j-1}^-, a_j^+]\!], a_1^-]\!]_{q^{(1+(-1)^{\theta_{j-1}}\delta_{j-1,1})}} = 0$ according to (38d).

(iii) In a similar way one shows that $[\![e_i, f_1]\!] = 0$ for $i > 1$.

(iv) The case $i, j \in [2; n+m]$. From (37)

$[\![e_i, f_j]\!] = (-1)^{\theta_{i-1,j-1}}[\![[\![a_i^-, a_{i-1}^+]\!]L_{i-1}, \bar{L}_{j-1}[\![a_{j-1}^-, a_j^+]\!]]\!]$.

Apply (39):

$$\begin{aligned}[\![e_i, f_j]\!] = {} & (-1)^{\theta_{i-1,j-1}} q^{(-1)^{\theta_{i-1}} - (-1)^{\theta_{j-1}}\delta_{ij} + (-1)^{\theta_{j-1}}\delta_{i,j-1}} L_{i-1}\bar{L}_{j-1} \\ & \times [\![[\![a_i^-, a_{i-1}^+]\!], [\![a_{j-1}^-, a_j^+]\!]]\!]_{q^{(-1)^{\theta_{i-1}}\delta_{i-1,j} - (-1)^{\theta_{j-1}}\delta_{i,j-1}}}\end{aligned} \qquad (40)$$

(iv.1) For $i = j$ (40) reduces to

$$[\![e_i, f_i]\!] = [\![[\![a_i^-, a_{i-1}^+]\!], [\![a_{i-1}^-, a_i^+]\!]]\!]. \qquad (41)$$

In order to evaluate the r.h.s. of (41) use the following identity ($\alpha = deg(a),\ \beta = deg(b)$):

If $x = zs,\ y = zr,\ t = zsr;\quad x, y, z, r, s, t \in \mathbf{C}[[h]]$, then

$$[\![a, [\![b, c]\!]_x]\!]_y = [\![[\![a, b]\!]_z, c]\!]_t + z(-1)^{\alpha\beta}[\![b, [\![a, c]\!]_r]\!]_s. \qquad (42)$$

Applying (42) to the r.h.s. of (41) with $a = [\![a_{i-1}^+, a_i^-]\!]$, $b = a_i^+$, $c = a_{i-1}^-$ and $x = y = 1$, $z = q$, $r = s = t = \bar{q}$, one obtains

$[\![e_i, f_i]\!] = [\![[\![[\![a_{i-1}^+, a_i^-]\!], a_i^+]\!]_q, a_{i-1}^-]\!]_{\bar{q}} - q(-1)^{\theta_i}[\![a_i^+, [\![[\![a_i^-, a_{i-1}^+]\!], a_{i-1}^-]\!]_{\bar{q}}]\!]_{\bar{q}}]\!]$

(use (38d))

$= [\![\bar{L}_i a_{i-1}^+, a_{i-1}^-]\!]_{\bar{q}} - q(-1)^{\theta_{i,i-1}}[\![a_i^+, \bar{L}_{i-1}a_i^-]\!]_{\bar{q}}$

(use (39))
$= \bar{L}_i [\![a_{i-1}^+, a_{i-1}^-]\!] - (-1)^{\theta_{i,i-1}} \bar{L}_{i-1} [\![a_i^+, a_i^-]\!]$
(use ($38c$) and (34))
$= \frac{(-1)^{\theta_{i-1}}}{q-\bar{q}} \left(k_i^{(-1)^{\theta_{i-1}}} - k_i^{-(-1)^{\theta_{i-1}}} \right).$
Taking into account that $\theta_{i-1} = 0$, for $i \in [1; n+1]$ and that $\theta_{i-1} = 1$ for $i \in [n+2; n+m]$, one ends with

$$[\![e_i, f_i]\!] = \frac{k_i - \bar{k}_i}{q - \bar{q}}. \tag{43}$$

(iv.2) Let $|i-j| > 1$. Then (40) reduces to
$[\![e_i, f_j]\!] = (-1)^{\theta_{i-1,j-1}} q^{(-1)^{\theta_{i-1}}} L_{i-1} \bar{L}_{j-1} [\![[\![a_i^-, a_{i-1}^+]\!], [\![a_{j-1}^-, a_j^+]\!]]\!]$
$= (-1)^{\theta_{ij}} q^{(-1)^{\theta_{i-1}}} L_{i-1} \bar{L}_{j-1} [\![[\![a_{i-1}^+, a_i^-]\!], [\![a_j^+, a_{j-1}^-]\!]]\!]$
(apply (42) with $a = [\![a_{i-1}^+, a_i^-]\!]$, $b = a_j^+$, $c = a_{j-1}^-$, $x = y = 1$, $z = q$, $t = s = r = \bar{q}$)
$= (-1)^{\theta_{ij}} q^{(-1)^{\theta_{i-1}}} L_{i-1} \bar{L}_{j-1} \big([\![[\![[\![a_{i-1}^+, a_i^-]\!], a_j^+]\!]_q, a_{j-1}^-]\!]_{\bar{q}}$
$-q(-1)^{\theta_{i,i-1}\theta_j + \theta_{i,i-1}} [\![a_j^+, [\![[\![a_i^-, a_{i-1}^+]\!], a_{j-1}^-]\!]_{\bar{q}}]\!]_{\bar{q}} \big) = 0$ from ($38d$).
(iv.3) For $j = i-1$ (40) reduces to
$[\![e_i, f_{i-1}]\!] = (-1)^{\theta_{i-1,i-2}} q^{(-1)^{\theta_{i-1}}} L_{i-1} \bar{L}_{i-2} [\![[\![a_{i-1}^+, a_i^-]\!], [\![a_{i-1}^+, a_{i-2}^-]\!]]\!]_{q^{(-1)^{\theta_{i-1}}}}$
(use (42) with $a = [\![a_{i-1}^+, a_i^-]\!]$, $b = a_{i-1}^+$, $c = a_{i-2}^-$, $x = 1$, $y = q^{(-1)^{\theta_{i-1}}}$, $r = t = \bar{q}$, $z = q^{1+(-1)^{\theta_{i-1}}}$, $s = q^{-(1+(-1)^{\theta_{i-1}})}$)
$= (-1)^{\theta_{i-1,i-2}} q^{(-1)^{\theta_{i-1}}} L_{i-1} \bar{L}_{i-2} ([\![[\![[\![a_{i-1}^+, a_i^-]\!], a_{i-1}^+]\!]_{q^{1+(-1)^{\theta_{i-1}}}}, a_{i-2}^-]\!]_{\bar{q}}$
$+q^{1+(-1)^{\theta_{i-1}}} [\![a_{i-1}^+, [\![[\![a_{i-1}^+, a_i^-]\!], a_{i-2}^-]\!]_{\bar{q}}]\!]_{q^{-(1+(-1)^{\theta_{i-1}})}}) = 0$ from ($38d$)
(iv.4) Let $j = i+1$. Then (40) reduces to
$[\![e_i, f_{i+1}]\!] = (-1)^{\theta_{i-1,i}} q^{(-1)^{\theta_{i-1}} + (-1)^{\theta_i}} L_{i-1} \bar{L}_i [\![[\![a_i^-, a_{i-1}^+]\!], [\![a_i^-, a_{i+1}^+]\!]]\!]_{q^{-(-1)^{\theta_i}}}$
$= (-1)^{\theta_{i,i+1}} q^{(-1)^{\theta_{i-1}} + (-1)^{\theta_i}} L_{i-1} \bar{L}_i [\![[\![a_{i-1}^+, a_i^-]\!], [\![a_{i+1}^+, a_i^-]\!]]\!]_{q^{-(-1)^{\theta_i}}}$
(from (42) with $a = [\![a_{i-1}^+, a_i^-]\!]$, $b = a_{i+1}^+$, $c = a_i^-$, $x = 1$, $y = q^{-(-1)^{\theta_i}}$, $z = q$, $s = \bar{q}$, $r = t = q^{-(1+(-1)^{\theta_i})}$)
$= (-1)^{\theta_{i,i+1}} q^{(-1)^{\theta_{i-1}} + (-1)^{\theta_i}} L_{i-1} \bar{L}_i ([\![[\![[\![a_{i-1}^+, a_i^-]\!], a_{i+1}^+]\!]_q, a_i^-]\!]_{q^{-(1+(-1)^{\theta_i})}}$
$+q(-1)^{\theta_{i-1,i}} [\![a_{i+1}^+, [\![[\![a_{i-1}^+, a_i^-]\!], a_i^-]\!]_{q^{-(1+(-1)^{\theta_i})}}]\!]_{\bar{q}}) = 0$, according to ($38d$)).

So far we have shown that all Cartan-Kac relations (21) follow from (38). It remains to verify the Serre relations (22). We consider in some more details the $e-$Serre relations ($22a-c$).

D) We pass to prove first ($22a$), namely that $[\![e_i, e_j]\!] = 0$ if $|i-j| \neq 1$.

(i) The case with $i = 1$ and $j = [3; n+m]$ follows directly from (39) and ($38d$).

(ii) $i \neq j \in [2; n+m]$. From ($37a$)
$[e_i, e_j] = (-1)^{\theta_{i-1,j-1}} [[\![a_i^-, a_{i-1}^+]\!] L_{i-1}, [\![a_j^-, a_{j-1}^+]\!] L_{j-1}]$
(use (39))
$= (-1)^{\theta_{ij}} [[\![a_{i-1}^+, a_i^-]\!], [\![a_{j-1}^+, a_j^-]\!]] L_{i-1} L_{j-1}$
(apply (42) with $a = [\![a_{i-1}^+, a_i^-]\!]$, $b = a_{j-1}^+$, $c = a_j^-$, $x = y = 1$, $z = q$, $t = r = s = \bar{q}$)
$= (-1)^{\theta_{ij}} ([\![[\![[\![a_{i-1}^+, a_i^-]\!], a_{j-1}^+]\!]_q, a_j^-]\!]_{\bar{q}} + q(-1)^{\theta_{i,i-1}\theta_{j-1}} [\![a_{j-1}^+, [\![[\![a_{i-1}^+, a_i^-]\!], a_j^-]\!]_{\bar{q}}]\!]_{\bar{q}}) L_{i-1} L_{j-1} = 0$, according to ($38d$).

(iii) If $i = j \neq n+1$, $[\![e_i, e_i]\!] = [e_i, e_i] = 0$

(iv) Consider $e_{n+1}^2 = \frac{1}{2} \{ e_{n+1}, e_{n+1} \} \equiv \frac{1}{2} [\![e_{n+1}, e_{n+1}]\!]$.

(iv.1) The case with $n+1=1$ is evident: $\{e_1, e_1\} = \{a_1^-, a_1^-\} = 0$, see (38$e$).

(iv.2) $n+1 \neq 1$. Use (37a): $e_{n+1}^2 \sim \{e_{n+1}, e_{n+1}\}_{q^2} = \{[\![a_{n+1}^-, a_n^+]\!]L_n, [\![a_{n+1}^-, a_n^+]\!]L_n\}_{q^2}$

$= \bar{q}[\![[\![a_{n+1}^-, a_n^+]\!], [\![a_{n+1}^-, a_n^+]\!]]\!]_{q^2} L_n^2$

(apply (42) with $a = [\![a_{n+1}^-, a_n^+]\!]$, $b = a_{n+1}^-$, $c = a_n^+$, $x = s = z = 1$, $y = r = t = q^2$)

$= \bar{q}([\![[\![[\![a_{n+1}^-, a_n^+]\!], a_{n+1}^-]\!], a_n^+]\!]_{q^2} - [\![a_{n+1}^-, [\![[\![a_{n+1}^-, a_n^+]\!], a_n^+]\!]_{q^2}]\!])L_n^2 = 0$, according to (38$d$)). Hence the Serre relations (22a) follow from (38).

E) We prove the triple Serre relation $[e_i, [e_i, e_{i+1}]_{\bar{q}}]_q = [e_i, [e_i, e_{i+1}]_q]_{\bar{q}} = 0, \quad i \neq n+1$.

(i) Let $i=1$. Since $n+1 \neq 1$, a_1^- is an even generator. Taking this into account, one easily derives only from (38) and (39) that

$[e_1, e_2]_{\bar{q}} = [a_1^-, [a_1^+, a_2^-]L_1]_{\bar{q}} = [a_1^-, [a_1^+, a_2^-]]_q L_1 = a_2^-$. Therefore, see (38$e$),

$[e_1, [e_1, e_2]_{\bar{q}}]_q = [a_1^-, a_2^-]_q = 0$.

(ii) $i \in [2; n]$. From (37a) and (39) $[e_i, e_{i+1}]_{\bar{q}} = [[a_i^-, a_{i-1}^+]L_{i-1}, [a_{i+1}^-, a_i^+]L_i]_{\bar{q}}$

$= [[a_i^-, a_{i-1}^+], [a_{i+1}^-, a_i^+]]L_i L_{i-1}$

(apply (42) with $a = [a_i^-, a_{i-1}^+]$, $b = a_{i+1}^-$, $c = a_i^+$, $x = y = 1$, $z = \bar{q}$, $r = s = t = q$)

$= [[[a_i^-, a_{i-1}^+], a_{i+1}^-]_{\bar{q}}, a_i^+]_q L_i L_{i-1} + \bar{q}[a_{i+1}^-, [[a_i^-, a_{i-1}^+], a_i^+]_q]_q L_i L_{i-1}$

(use (38d) and (39))

$= -\bar{q}[a_{i+1}^-, \bar{L}_i a_{i-1}^+]_q L_i L_{i-1} = -[a_{i+1}^-, a_{i-1}^+]L_{i-1}$.

Therefore

$[e_i, [e_i, e_{i+1}]_{\bar{q}}]_q = [[a_i^-, a_{i-1}^+]L_{i-1}, [a_{i+1}^-, a_{i-1}^+]L_{i-1}]_q = \bar{q}[[a_i^-, a_{i-1}^+], [a_{i+1}^-, a_{i-1}^+]]_q L_{i-1}^2$

(from (42) with $a = [a_i^-, a_{i-1}^+]$, $b = a_{i+1}^-$, $c = a_{i-1}^+$, $x = 1$, $y = s = q$, $z = \bar{q}$, $r = t = q^2$)

$= \bar{q}([[[a_i^-, a_{i-1}^+], a_{i+1}^-]_{\bar{q}}, a_{i-1}^+]_{q^2}]_q + \bar{q}[a_{i+1}^-, [[a_i^-, a_{i-1}^+], a_{i-1}^+]_{q^2}]_q)L_{i-1}^2 = 0$, according to (38$d$).

(iii) $i \in [n+2; n+m]$. Again evaluate first

$[e_i, e_{i+1}]_q = [[\![a_i^-, a_{i-1}^+]\!], [\![a_{i+1}^-, a_i^+]\!]]L_i L_{i-1}$

(from (42) with $a = [\![a_i^-, a_{i-1}^+]\!]$, $b = a_{i+1}^-$, $c = a_i^+$, $x = y = 1$, $z = \bar{q}$, $r = s = t = q$)

$= [\![[\![[\![a_i^-, a_{i-1}^+]\!], a_{i+1}^-]\!]_{\bar{q}}, a_i^+]\!]_q L_i L_{i-1} + \bar{q}[\![a_{i+1}^-, [\![[\![a_i^-, a_{i-1}^+]\!], a_i^+]\!]_q]\!]_q L_i L_{i-1}$

(use (38d))

$= \bar{q}[\![a_{i+1}^-, \bar{L}_i a_{i-1}^+]\!] = [\![a_{i+1}^-, a_{i-1}^+]\!]L_{i-1}$.

Hence

$[e_i, [e_i, e_{i+1}]_q]_{\bar{q}} = [[\![a_i^-, a_{i-1}^+]\!]L_{i-1}, [\![a_{i+1}^-, a_{i-1}^+]\!]L_{i-1}]_{\bar{q}} = q[[\![a_i^-, a_{i-1}^+]\!], [\![a_{i+1}^-, a_{i-1}^+]\!]]_{\bar{q}} L_{i-1}^2$

(from (42) with $a = [\![a_i^-, a_{i-1}^+]\!]$, $b = a_{i+1}^-$, $c = a_{i-1}^+$, $x = r = t = 1$, $y = z = \bar{q}$, $s = q$ and the triple relations (38d))

$= 0$.

The other triple e-Serre relation $[e_i, [e_i, e_{i-1}]_{\bar{q}}]_q = [e_i, [e_i, e_{i-1}]_q]_{\bar{q}} = 0$ is proved in a similar way.

F) In order to complete the proof it is convenient to show as an intermediate step that Eqs. (28) are consequence of (38). We begin with the l.h.s. of (28a).

$[\![e_i, a_j^-]\!]_{q_j^{\delta_{i-1,j}-\delta_{ij}}} = [\![-(-1)^{\theta_{i-1}}[\![a_i^-, a_{i-1}^+]\!]L_{i-1}, a_j^-]\!]_{q_j^{\delta_{i-1,j}-\delta_{ij}}}$. Push L_{i-1} to the right and expand the outer supercommutator:

$$\begin{aligned}&[\![e_i, a_j^-]\!]_{q_j^{\delta_{i-1,j}-\delta_{ij}}} = -(-1)^{\theta_{i-1}}\big(q^{1+(-1)^{\theta_{i-1}}\delta_{i-1,j}}[\![a_i^-, a_{i-1}^+]\!]a_j^- \\ &- (-1)^{\theta_{i,i-1}\theta_j} q_j^{\delta_{i-1,j}-\delta_{ij}} a_j^-[\![a_i^-, a_{i-1}^+]\!]\big)L_{i-1}.\end{aligned} \tag{44}$$

(i) The case $j < i-1$. From (44) $[\![e_i, a_j^-]\!] = -(-1)^{\theta_{i-1}} q[\![[\![a_i^-, a_{i-1}^+]\!], a_j^-]\!]_{\bar q} L_{i-1} = 0$, according to ($38d$), i.e., ($28a$) holds for $j < i-1$.

(ii) The case $j = i-1$. From (44)

$$[\![e_i, a_{i-1}^-]\!]_{q_{i-1}} = -(-1)^{\theta_{i-1}}\left(q^{1+(-1)^{\theta_{i-1}}}[\![a_i^-, a_{i-1}^+]\!]a_{i-1}^- - q_{i-1}a_{i-1}^-[\![a_i^-, a_{i-1}^+]\!]\right)L_{i-1}. \tag{45}$$

(ii.1) If $i \in [1; n+1]$, then $\theta_{i-1} = 0$, $q_{i-1} = q$ and (45) & ($38d$) yield
$[\![e_i, a_{i-1}^-]\!]_q = -q^2[\![[\![a_i^-, a_{i-1}^+]\!], a_{i-1}^-]\!]_{\bar q} L_{i-1} = -q a_i^-$.

(ii.2) If $i \in [n+2; n+m]$, then $\theta_{i-1} = 1$, $q_{i-1} = \bar q$ and (45) & ($38d$) yield
$[\![e_i, a_{i-1}^-]\!]_{\bar q} = [\![[\![a_i^-, a_{i-1}^+]\!], a_{i-1}^-]\!]_{\bar q} L_{i-1} = -\bar q a_i^-$. Hence for $j = i-1$ ($28a$) is fulfilled.

(iii) The case $j = i$. Then (44) reduces to

$$[\![e_i, a_i^-]\!]_{\bar q_i} = -(-1)^{\theta_{i-1}}\left(q[\![a_i^-, a_{i-1}^+]\!]a_i^- - \bar q_i a_i^-[\![a_i^-, a_{i-1}^+]\!]\right)L_{i-1}. \tag{46}$$

(iii.1) If $i \in [1; n+1]$, then $\theta_{i-1} = 0$, $q_i = q^{(-1)^{\theta_i}}$ and (46) & ($38d$) yield
$[\![e_i, a_i^-]\!]_{q^{-(-1)^{\theta_i}}} = -q[\![[\![a_i^-, a_{i-1}^+]\!], a_i^-]\!]_{q^{-(1+(-1)^{\theta_i})}} L_{i-1} = 0$.

(iii.2) If $i \in [n+2; n+m]$, then $\theta_{i-1} = 1$, $q_i = \bar q$ and (46) & ($38d$) yield
$[\![e_i, a_i^-]\!]_q = q[\![[\![a_i^-, a_{i-1}^+]\!], a_i^-]\!] L_{i-1} = 0$.
Hence for $i = j$ ($28a$) is fulfilled.

(iv) The case $j > i$. Then (44) & ($38d$) yield
$[\![e_i, a_j^-]\!] = -(-1)^{\theta_{i-1}} q[\![[\![a_i^-, a_{i-1}^+]\!], a_j^-]\!]_{\bar q} L_{i-1} = 0$.
Therefore ($28a$) is a consequence of Eqs. (38).

In a similar way one proves that the other relations (28) can be derived from Eqs. (38).

Note that from Eqs. (28) one derives also Eqs. ($24a, b$).

G) We are ready now to derive the additional Serre relation ($22c$).

Using ($24a$), write $a_{n+2}^- = [[[a_{n-1}^-, e_n]_{\bar q}, e_{n+1}]_{\bar q}, e_{n+1}]_q$. From ($28a$) $\{e_{n+1}, a_{n+2}^-\} = 0$. Therefore $0 = \{e_{n+1}, a_{n+2}^-\} = \{e_{n+1}, [[[a_{n-1}^-, e_n]_{\bar q}, e_{n+1}]_{\bar q}, e_{n+1}]_q\}$. Since $[e_{n+1}, a_{n-1}^-] = 0$, and $[e_{n+2}, a_{n-1}^-] = 0$ (see ($28a$)) applying twice (29) and once (30) one obtains $0 = \{e_{n+1}, a_{n+2}^-\} = [a_{n-1}^-, y]_{\bar q}$, where

$$y = \{e_{n+1}, [[e_n, e_{n+1}]_{\bar q}, e_{n+2}]_q\}. \tag{47}$$

Therefore $[y, a_{n-1}^-]_q = 0$. From ($24b$), ($21c$) and (47) it follows that $[y, a_{n-1}^+] = 0$. Applying (29) we have
$0 = [[y, a_{n-1}^-]_q, a_{n-1}^+] = [y, [a_{n-1}^-, a_{n-1}^+]]_q$
(use ($38c$), (24c) and ($21b$))
$= (q - \bar q)^{-1}[y, L_{n-1} - \bar L_{n-1}]_q = qy\bar L_{n-1}$. Hence, $y = 0$, i.e., the additional $e-$Serre relation ($22c$) holds.

H) In a similar way one derives the $f-$Serre relations ($22d - f$). Another way to prove them is to apply the star-operation on the $e-$ Serre relations.

This completes the proof of the Theorem.

4. Discussions and further outlook

In the present paper we enlarge the list of the quantum superalgebras, which admit a description via deformed creation and annihilation generators [8-13], adding to it all quantum superalgebras $U_q[sl(n+1|m)]$. The possibility for such a description is not unexpected. We have generalized the results for $U_q[sl(n+1)]$ [13] to the superalgebra case. This generalization is however, we wish to point out, neither evident nor straightforward. The "super" structure is richer, with more relations ($e_{n+1}^2 = f_{n+1}^2 = 0$, additional Serre relations $(22c, f)$) and, as a result, with several features which do not appear in the Lie algebra cases (the simple root systems are not related by transformations from the Weyl group, one and the same superalgebra admits several Dynkin diagrams, etc.). All these peculiarities, especially in the deformed case, which we have mainly in mind here and bellow, make the computations nontrivial, technically much more involved.

In the introduction we said few words for a justification of the name *creation and annihilation generators*. Another reason for this name stems from the observation that, using the CAGs, one can construct Fock spaces in a much similar way as in the parastatistics quantum field theory (postulating the existence of a vacuum, which is annihilated by all a_i^- operators and introducing an order of the statistics [16]; for more details on parastatistics see, for instance, [32]). Then the Fock spaces are generated by the creation operators, acting on the vacuum. Moreover a_i^+, acting on a state with fixed number of "particles" (elementary excitation) of species i, increases them by one, whereas a_i^- diminishes them by one. The advantage of this property for the physical applications and interpretation is evident. Consider, for instance, a "free" Hamiltonian

$$H = \sum_{i=1}^{n+m} \varepsilon_i H_i, \quad \text{such that} \quad \sum_{i=1}^{n+m} (-1)^{\theta_i} \varepsilon_i = 0, \tag{48}$$

which in the nondeformed case takes the usual form

$$H = \sum_{i=1}^{n+m} \varepsilon_i [\![a_i^+, a_i^-]\!]. \tag{49}$$

Then

$$[H, a_i^\pm] = \pm \varepsilon_i a_i^\pm, \tag{50}$$

i.e., a_i^+ (resp. a_i^-) can be interpreted as an operator creating (resp. annihilating) a "particle" of species i with energy ε_i. Our, we call it *physical conjecture* is that the Fock representations of the deformed CAGs will lead to new solutions for the microscopic $g-$ons statistics in the sense of Karabali and Nair [33], which is a particular realization of the exclusion statistics of Haldane [27].

The Fock representations however may be of interest also from another, more mathematical point of view. So far the finite-dimensional irreducible representations of the LSs from the class A were explicitly constructed only for $sl(n|1)$ [34]. Any such representation can be deformed to a representation of $U_q[sl(n|1)]$ [35]. The representation theory of $sl(n|m)$, $n, m = 1, 2, \ldots$ and hence of the corresponding deformed algebras is however far from being complete, if both $n \neq 1$ and $m \neq 1$. In [36] the so-called essentially typical representations of $sl(n|m)$ were described. The results were generalized also to the quantum case [37]. Our *mathematical conjecture* now is that the Fock representations are beyond the class of the deformed essentially typical representation [36], thus yielding new representations of $U_q[sl(n+1|m)]$.

In order to verify the above conjectures one would need to construct the Fock representations explicitly, i.e., to introduce a basis and to write down the transformations of the basis under the action of the generators. As a first step one has to determine the quantum analogue of the triple relations (17). This is a nontrivial problem. It actually means that one has to write down the supercommutation relations between all Cartan-Weyl generators, expressed via the CAGs. The latter is a necessary condition for the application of the Poincare-Birghoff-Witt theorem, when computing the action of the generators on the Fock basis vectors. We return to this problem elsewhere. Here we mention only one, but important additional relation: from (17) one derives that the creation (resp. annihilation) generators q-supercommute,

$$[\![a_i^\xi, a_j^\xi]\!]_{q'} = 0, \quad q' = q \text{ or } \bar{q}, \;\; i,j \in [1; n+m], \;\; \xi = \pm. \tag{51}$$

This makes evident the basis (or at least one possible basis) in a given Fock space, since any product of only creation generators can be always ordered. Note that similar property does not hold for para-Bose (or para-Fermi) creation operators. This is the reason why (even in the nondeformed case) the matrix elements of the paraoperators remain still unknown for an arbitrary order of the parastatistics: the Fock space basis cannot be represented as ordered products of only para-Bose (or para-Fermi) creation operators acting on the vacuum (the linear span of only such vectors is not invariant under the action of the para-operators).

Finally let us mention that we do not have simple relations for the action of the other Hopf algebra operations (Δ, ε, S) on the CAGs, although it is clear how to write them down, using Eqs. (16) and the circumstance that the comultiplication Δ and the counity ε are morphisms, whereas the antipode S is an antimorphism. In this respect the picture is much the same as discussed in [13]. Luckily, the (Δ, ε, S)-operations are not necessary for computing the transformations of the Fock modules (but they are certainly very important when considering tensor products of representation spaces).

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